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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2024
General Certificate of Education (Adv. Level) Examination, 2024

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

✻ Answer **five** questions only.

11. (*a*) Let $f(x) = x^2 + 2x + c$, where $c \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $c < 1$.

Let α and β be the roots of $f(x) = 0$.

Show that $\alpha^2 + \beta^2 = 4 - 2c$.

Let $c \neq 0$ and $\lambda \in \mathbb{R}$. The quadratic equation with $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ as its roots is $2x^2 + 12x + \lambda = 0$. Find the values of c and λ.

(*b*) Let $f(x) = x^3 + px^2 + qx + p$, where $p, q \in \mathbb{R}$. The remainder when $f(x)$ is divided by $(x - 2)$ is 36 more than the remainder when $f(x)$ is divided by $(x - 1)$. Show that $3p + q = 29$.

It is also given that $(x + 1)$ is a factor of $f(x)$. Show that $p = 6$ and $q = 11$, and factorize $f(x)$ completely.

Hence, solve $f(x) = 3(x + 2)$.

12. (*a*) The parents of a family decide to invite 6 out of 15 of their close relatives for a dinner. While the father has 5 close female relatives and 3 close male relatives, the mother has 3 close female relatives and 4 close male relatives.

Find the number of different ways in which

(i) the father can invite 3 of his close female relatives and the mother can invite 3 of her close male relatives,

(ii) the father can invite 3 of his close relatives and the mother can invite 3 of her close relatives so that 3 males and 3 females are invited.

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(*b*) Let $U_r = \dfrac{1}{r(r+2)(r+4)}$ and $f(r) = \dfrac{1}{r(r+2)}$ for $r \in \mathbb{Z}^+$.

Determine the value of the real constant A such that $f(r) - f(r+2) = AU_r$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} U_r = \dfrac{11}{96} - \dfrac{1}{4(n+1)(n+3)} - \dfrac{1}{4(n+2)(n+4)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Find the value of the real constant m such that $\lim_{n\to\infty} \sum_{r=1}^{n} (mU_r + U_{n+1-r}) = \dfrac{11}{32}$.

13. (*a*) Let $a, b \in \mathbb{R}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & a & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & a & b \\ 3 & b & a \end{pmatrix}$. It is given that $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$.

Show that $a = 0$ and $b = 5$.

With these values for a and b, let $\mathbf{C} = \mathbf{A}\mathbf{B}^{\mathrm{T}}$.

Find $\mathbf{C}$ and write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{D}$ such that $\mathbf{DC} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(*b*) Let $z_1, z_2 \in \mathbb{C}$. Show that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $z_1 \overline{z_1} = |z_1|^2$

Using the result that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ for $z_2 \neq 0$, show that if $|z_1| = 1$ and $z_1 \neq \pm 1$, and also if $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ is real, then $|z_2| = 1$.

(*c*) Express $\sqrt{3} + i$ is the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$.

Using De Moivre's theorem, show that $\dfrac{(\sqrt{3}+i)^{24}}{2^{23}(1+i)} = 1 - i$.

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14. (*a*) Let $f(x)=\dfrac{px+q}{(x-1)(x-2)}$ for $x \in \mathbb{R}-\{1, 2\}$, where $p, q \in \mathbb{R}$. It is given that the graph of $y=f(x)$ has a stationary point at $(0, 1)$. Show that $p=-3$ and $q=2$.

For these values of p and q, show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$, and find the intervals on which $f(x)$ is decreasing and the intervals on which $f(x)$ is increasing.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Hence, find the number of real solutions to the equation $x^2(x-1)(x-2)=2-3x$.

(*b*) A cylinder with a top and a bottom is made to have a volume of 1024π cm^3. Let r cm be the radius of the cylinder. Show that the total surface area S cm^2 of the cylinder is given by $S=2\pi\left(\dfrac{1024}{r}+r^2\right)$ for $r>0$.

Show that S is minimum when $r=8$.

15. (*a*) Find the values of the real constants A and B such that $3t^2+4=A(t^2-2t+4)+Bt(t+1)$ for all $t \in \mathbb{R}$.

Hence or otherwise, find $\displaystyle\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,dt$.

(*b*) Using the substitution $u=x+\sqrt{x^2+3}$, show that $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+3}}\,dx=\frac{1}{2}\ln 3$.

Let $J=\displaystyle\int_0^1 \sqrt{x^2+3}\,dx$. Using integration by parts, show that $2J=2+\displaystyle\int_0^1 \frac{3}{\sqrt{x^2+3}}\,dx$.

Deduce that $J=1+\dfrac{3}{4}\ln 3$.

(*c*) Using the formula $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, where a is a constant, show that

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x+\sin x}\right)dx=\frac{\pi}{8}\ln\left(\frac{1}{2}\right).$$

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16. Let $A \equiv (1, 2)$ and $B \equiv (a, b)$, where $a, b \in \mathbb{R}$. It is given that the perpendicular bisector l of the line segment AB has the equation $x + y - 4 = 0$. Find the values of a and b.

Let $C \equiv (3, 1)$. Show that the point C lies on the line l and find $A\hat{C}B$.

Let S be the circle through the points A, B and C. Show that the centre of S is given by $\left(\frac{13}{6}, \frac{11}{6}\right)$ and find the equation of S.

Hence, find the equation of the circle passing through the points A, B and the point $D \equiv (0, 3)$.

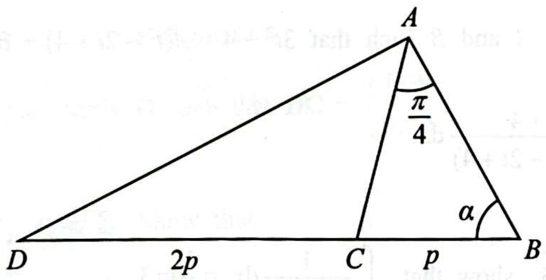
17. (*a*) Express $6\cos 2x - 8\sin 2x$ in the form $R\cos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve $6\cos 2x - 8\sin 2x = 5$.

Express $24\cos^2 x - 32\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$, where a, b, c $(\in \mathbb{R})$ are constants to be determined.

Deduce the minimum value of $24\cos^2 x - 32\sin x\cos x$.

(*b*)



In the triangle ABC shown in the figure, $BC = p$, $B\hat{A}C = \frac{\pi}{4}$ and $A\hat{B}C = \alpha$. The point D lies on the extended line BC such that $CD = 2p$.

Show that $AB = p(\cos\alpha + \sin\alpha)$.

Find AD^2 in terms of p and α.

Deduce that if $AD = 3p$, then $\alpha = \tan^{-1}(5)$.

(*c*) Solve the equation $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$.

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