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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number							

Instructions:

- ※ *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
- ※ **Part A:** *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
- ※ **Part B:** *Answer **five** questions only. Write your answers on the sheets provided.*
- ※ *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
- ※ *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	

Total

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(6r+1) = n(3n+4)$ for all $n \in \mathbb{Z}^{+}$.

2. Sketch the graphs of $y = 2|x+1|$ and $y = 2-|x|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $2|x+2|+|x| \leq 4$.

[see page three

Index Number | | | | | | | |

3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\mathrm{Arg}(z-1-i) = -\frac{\pi}{4}$.

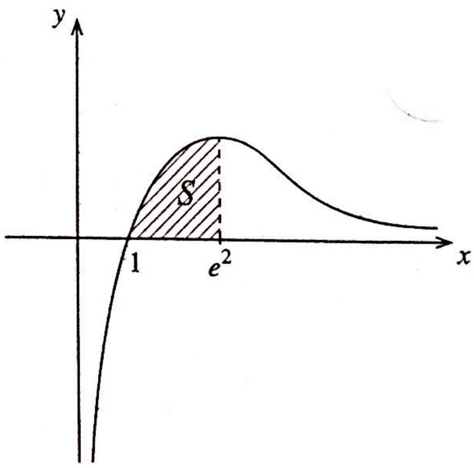
Hence or otherwise, show that the minimum value of $|z-2+i|$ satisfying $\mathrm{Arg}(iz+1-i) = \frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$.

4. Let $k > 0$. It is given that the coefficient of x^7 in the binomial expansion of $\left(x^2+\frac{k}{x}\right)^{11}$ and the coefficient of x^{-7} in the binomial expansion of $\left(x-\frac{1}{x^2}\right)^{11}$ are equal. Show that $k = 1$.

[see page four

5. Show that $\lim_{x \to 0} \dfrac{\tan 2x - \sin 2x}{x^2\left(\sqrt{1+x} - \sqrt{1-x}\right)} = 4$.

6. Let S be the region enclosed by the curves $y = \dfrac{\ln x}{\sqrt{x}}$, $y = 0$ and $x = e^2$. Show that the area of S is 4 square units.
The region S is rotated about the x–axis through 2π radians. Show that the volume of the solid thus generated is $\dfrac{8\pi}{3}$.



[see page five

7. Show that the equation of the tangent line to the rectangular hyperbola parametrically given by $x = ct$ and $y = \frac{c}{t}$ for $t \neq 0$, at the point $P \equiv \left(cp, \frac{c}{p}\right)$ is given by $x + p^2y = 2cp$.

The normal line to this hyperbola at P meets the hyperbola again at another point $Q \equiv \left(cq, \frac{c}{q}\right)$.

Show that $p^3q = -1$.

8. Let $A \equiv (0, -1)$ and $B \equiv (9, 8)$. The point C lies on AB such that $AC:CB = 1:2$. Show that the equation of the straight line l through C perpendicular to AB is $x + y - 5 = 0$.

Let D be the point on l such that AD is parallel to the straight line $y = 5x + 1$. Find the coordinates of D.

[see page six

9. Show that the straight line $x + 2y = 3$ intersects the circle $S \equiv x^2 + y^2 - 4x + 1 = 0$ at two distinct points.

Find the equation of the circle passing through these two points and the centre of the circle $S = 0$.

10. Express $2\cos^2 x + 2\sqrt{3}\sin x \cos x - 1$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve the equation $\cos^2 x + \sqrt{3}\sin x \cos x = 1$.

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10 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let $k>1$. Show that the equation $x^2-2(k+1)x+(k-3)^2=0$ has real distinct roots.
Let α and β be these roots. Write down $\alpha+\beta$ and $\alpha\beta$ in terms of k, and find the values of k such that both α and β are positive.
Now, let $1<k<3$. Find the quadratic equation whose roots are $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\sqrt{\beta}}$, in terms of k.

(*b*) Let $f(x)=2x^3+ax^2+bx+1$ and $g(x)=x^3+cx^2+ax+1$, where $a, b, c\in\mathbb{R}$. It is given that the remainder when $f(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by x^2+x-2 is $x+1$. Find the values of a, b and c.
Also, with these values for a, b and c, show that $f(x)-2g(x)\le\frac{13}{12}$ for all $x\in\mathbb{R}$.

12. (*a*) It is required to form a 4-digit number consisting of 4 digits taken from the 10 digits given below:

1, 1, 1, 2, 2, 3, 3, 4, 5, 5

Find the number of different such 4-digit numbers that can be formed

(i) if all 4 digits chosen are different,

(ii) if any 4 digits can be chosen.

(*b*) Let $U_r=\dfrac{-16r^3+12r^2+40r+9}{5(2r+1)^2(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Determine the values of the real constants A and B such that $U_r=\dfrac{A(r-1)}{(2r+1)^2}-\dfrac{(r-B)}{(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Hence, find $f(r)$ such that $\frac{1}{5^{r-1}}U_r=f(r)-f(r-1)$ for $r\in\mathbb{Z}^+$, and

show that $\displaystyle\sum_{r=1}^{n}\frac{1}{5^{r-1}}U_r=1+\frac{n-1}{5^n(2n+1)^2}$ for $n\in\mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{5^{r-1}}U_r$ is convergent and find its sum.

[see page eight

13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 3 \\ 0 & a & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, where $a \in \mathbb{R}$.

Also, let $\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$. Find $\mathbf{C}$ in terms of a, and show that $\mathbf{C}^{-1}$ exists for all $a \neq 0$.

Write down $\mathbf{C}^{-1}$ in terms of a, when it exists.

Show that if $\mathbf{C}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 9 \\ -11 \end{pmatrix}$, then $a = 2$.

With this value for a, find the matrix $\mathbf{D}$ such that $\mathbf{DC} - \mathbf{C}^{\mathrm{T}}\mathbf{C} = 8\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z_1 = 1 + \sqrt{3}i$ and $z_2 = 1 + i$. Express $\frac{z_1}{z_2}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

Also, express each of the complex numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, and **hence**, show that $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.

Deduce that $\cos\left(\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

(*c*) Let $n \in \mathbb{Z}^+$ and $\theta \neq 2k\pi \pm \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

Using De Moivre's theorem, show that $(1 + i\tan\theta)^n = \sec^n\theta(\cos n\theta + i\sin n\theta)$.

Hence, obtain a similar expression for $(1 - i\tan\theta)^n$, and

show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = 2\sec^n\theta\cos n\theta$.

Deduce that $z = i\tan\left(\frac{\pi}{10}\right)$ is a solution of $(1+z)^{25} + (1-z)^{25} = 0$.

14. (*a*) Let $f(x) = \frac{4x+1}{x(x-2)}$ for $x \neq 0, 2$.

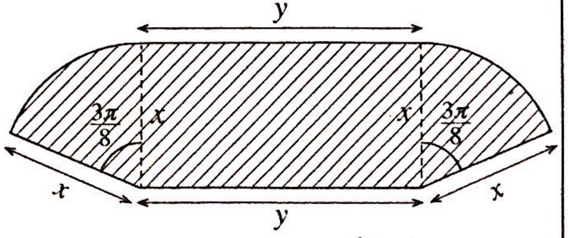
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\frac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0, 2$.

Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Sketch the graph of $y = f(x)$ indicating the asymptotes, x-intercept and the turning points.

Using this graph, find all real values of x satisfying the inequality $f(x) + |f(x)| > 0$.

(*b*) The shaded region S of the adjoining figure shows a garden consisting of a rectangle and two sectors of a circle each subtending an angle $\frac{3\pi}{8}$ at the centre. Its dimensions, in metres, are shown in the figure. The area of S is given to be $36\ \mathrm{m}^2$. Show that the perimeter p m of S is given by $p = 2x + \frac{72}{x}$ for $x > 0$ and that p is minimum when $x = 6$.



[see page nine

15.(*a*) Find the values of the constants A, B and C such that

$$x^4 + 3x^3 + 4x^2 + 3x + 1 = A(x^2+1)^2 + Bx(x^2+1) + Cx^2 \text{ for all } x \in \mathbb{R}.$$

Hence, write down $\dfrac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}$ in partial fractions and

find $\displaystyle\int \frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}\,dx$.

(*b*) Let $I = \displaystyle\int_0^{\frac{1}{4}} \sin^{-1}\left(\sqrt{x}\right)dx$. Show that $I = \dfrac{\pi}{24} - \dfrac{1}{2}\displaystyle\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-x}}\,dx$ and **hence**, evaluate I.

(*c*) Show that $\dfrac{d}{dx}\left(x\ln(x^2+1) + 2\tan^{-1}x - 2x\right) = \ln(x^2+1)$.

Hence, find $\displaystyle\int \ln(x^2+1)\,dx$ and show that $\displaystyle\int_0^1 \ln(x^2+1)\,dx = \frac{1}{2}(\ln 4 + \pi - 4)$.

Using the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant,

find the value of $\displaystyle\int_0^1 \ln\left[(x^2+1)(x^2-2x+2)\right]dx$.

16. Let $P \equiv (x_1, y_1)$ and l be the straight line given by $ax+by+c=0$. Show that the coordinates of any point on the line through the point P and perpendicular to l are given by (x_1+at, y_1+bt), where $t \in \mathbb{R}$.

Deduce that the perpendicular distance from P to l is $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

Let l be the straight line $x+y-2=0$. Show that the points $A \equiv (0,6)$ and $B \equiv (3,-3)$ lie on opposite sides of l.

Find the acute angle between l and the line AB.

Find the equations of the circles S_1 and S_2 with centres at A and B, respectively, and touching l.

Let C be the point of intersection of l and the line AB. Find the coordinates of the point C.

Find also the equation of the other common tangent through C to S_1 and S_2.

Show that the equation of the circle that passes through the origin, bisects the circumference of S_1 and orthogonal to S_2 is $3x^2+3y^2-38x-22y=0$.

[see page ten

17. (*a*) Write down $\cos(A+B)$ and $\cos(A-B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.

Hence, show that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Deduce that $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

Solve the equation $\cos 9x + \cos 7x + \cot x(\cos 9x - \cos 7x) = 0$.

(*b*) In the usual notation, state and prove the **Cosine Rule** for a triangle ABC.

Let $x \neq n\pi + \frac{\pi}{2}$ for $n \in \mathbb{Z}$. Show that $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$.

In a triangle ABC, it is given that $AB = 20$ cm, $BC = 10$ cm and $\sin 2B = \frac{24}{25}$.

Show that there are two distinct such triangles and find the length of AC for each.

(*c*) Solve the equation $\sin^{-1}\left[(1+e^{-2x})^{-\frac{1}{2}}\right] + \tan^{-1}(e^{x}) = \tan^{-1}(2)$.

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