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Western Province Educational Department

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
General Certificate of Education (Adv. Level) Examination, 2024

සංයුක්ත ගණිතය I
Combined Mathematics I

10 E I

2024.10.30 / 08.30 - 11.40

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Three hours

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Additional Reading Time - 10 minutes

additional reading time to go through the question paper, select the questions and decide on the questions that you give priority in answering.

Index Number							

Instructions:

* *This question paper consists of two parts.*
  Part A (Questions 01 - 10) *and* **Part B** (Questions 11 - 17)
* **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove only* **Part B** *of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics II		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

Part A

Answer for **all** questions.

01. Let $U_1 = 9$, and $U_{n+1} = 10U_n + 9$ for all $n \in \mathbb{Z}^+$. Using the **Principle of Mathematical Induction** prove that, $U_n = 10^n - 1$ for all $n \in \mathbb{Z}^+$.

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02. Sketch the graphs of $y = 2|x-3| - (x-3)$ and $y = 9 - |x|$ in the same diagram. **Hence or otherwise** find all real values of x satisfying the ineuality $2|x| + |x+3| > x + 9$

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03. Indicate the complex numbers z_1 and z_2 on an Argand diagram, such that $|z_1 - 2| = |z_1 - 2i|$ and $|z_2 - 2| = \sqrt{2}$. **Hence,** express the complex number z such that $z_1 = z_2$, in the form of $z = r(\cos\theta + i\sin\theta)$ Here, r and θ are the real constants to be determined.

04. If a committec consisting of 4 men and 3 women is to be formed from a group of 4 lady doctors, 5 engineers, 4 actresses and 3 male singers, in how many ways the committee can be formed ?
If this committee is going to take a photograph in such a way that, no two men are next to each other, then find the number of different photographs that can be taken.

05. Show that, $\lim_{x \to \frac{\pi}{2}} \left(\dfrac{\sqrt[3]{\sin x} - \dfrac{2x}{\pi}}{\pi^2 - 4x^2} \right) = \dfrac{1}{2\pi^2}$

06. Find the **area** of the region enclosed by the curves $y = |\cos x|$, $y = 0$, $4x - \pi = 0$, and $4x - 3\pi = 0$ If this region is totated about x axis through 2π radians, then show that the volume thus generated is $\frac{\pi}{4}(\pi - 2)$ cabic units.

07. Let $x = \text{cosec}\,\theta - \sin\theta$ and $y = \text{cosec}^n\theta - \sin^n\theta$ for a real parameter θ and a rational constant n.

Show that; $\left(x^2+4\right)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - n^2 y = 0$

08. Find the equation of the straight line l, so that the angle bisector of $l = 0$ and $x + 3y + 1 = 0$ is $x - y - 4 = 0$.

09. Show that the circle $x^2 + y^2 + 2x + 2y - 1 = 0$ and the straight line $x - y + 1 = 0$ intersects each other. If the new circle, which passes through those points of intersection, also goes through the origin, them find the **value of integer** n such that the area of this new circle is $\left(\frac{n+1}{n}\right)\pi$.

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10. Sketch the graph of $y = \cos x$ in the domain $x \in \left[0, \frac{\pi}{2}\right)$ **Hence,** when $\frac{\pi}{6} < \theta < \frac{\pi}{3}$, **deduce** that $\frac{3}{2\pi} < \frac{\cos\theta}{\theta} < \frac{3\sqrt{3}}{\pi}$

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Western Province Educational Department

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
General Certificate of Education (Adv. Level) Examination, 2024

සංයුක්ත ගණිතය I
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Part B

■ Answer **only five** Questions.

11. (*a*) Let's consider the quadratic equation $ax^2+bx+c=0$; for $a\neq 0$ and $a,\ b,\ c\in\mathbb{R}$. By using completing squares, show that $\dfrac{b\pm\sqrt{\Delta}}{-2a}$ are the roots of $ax^2+bx+c=0$. Here $\Delta=b^2-4ac$.

Hence, write down the necessary and sufficient condition for the quadratic equation $ax^2+bx+c=0$ to have only a pair of **rational roots,** when $a\neq 0$ and $a,\ b,\ c\in\mathbb{Q}$.

Now, let $F(x)=mx^3+(n-m)x^2-(m+2n)x+m+n$ for $m\neq 0$ and $m,\ n\in\mathbb{Q}$. Prove that 1 is a root of the cubic equation $F(x)=0$.

Also show that, all the roots of the cubic equation further, if the equation $F(x)=0$ has three real and equal (coincident) roots, deduce that $|m|:|n|=1:2$

(*b*) Let, H be a polynomial of order three. When $H(x)$ is seperately divided by (x^2+x-2) and (x^2+2x-3) the remainder is $x-6$.

Also, if it is given that the point $A\equiv(-1,\ -3)$ lies on the graph of $y=H(x)$ then find $H(x)$ and factorize it completely.

Further, write down the **domain** of H, so that all the images of the function H are non-negative.

12. (*a*) Find the **integers** A and B, such that, $A(2x+5)+B(2x+1)\equiv 4x+14$; for all $x\in\mathbb{R}$.

Now, write down the r^{th} term, U_r of the series, $\dfrac{1}{9}\cdot\dfrac{18}{3\cdot5\cdot7}+\dfrac{1}{27}\cdot\dfrac{22}{5\cdot7\cdot9}+\dfrac{1}{81}\cdot\dfrac{26}{7\cdot9\cdot11}+\cdots$ for $r\in\mathbb{Z}^+$

Hence, find $f(r)$, such that $U_r=f(r)-f(r+1)$; for $r\in\mathbb{Z}^+$ and show that,

$$\sum_{r=1}^{n}U_r=\frac{1}{45}-\frac{1}{3^{n+1}(2n+3)(2n+5)}\ ;\ \text{for } n\in\mathbb{Z}^+$$

Also, **deduce** that the infinite series $\sum_{r=1}^{\infty}U_r$ is convergent and find its sum.

Deduce for ther further that the sum of the infinite series $\sum_{r=2024}^{\infty}U_{r-2022}$.

(*b*) Expand $(x+y)^n$; for $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Also write down the $r+1$ term, T_{r+1} of above expansion. Now consider the expansion of $(1+x)^{23}$, so that teh powers of x are increasing. If the co-efficients of the terms including x^n, x^{n+1} and x^{n+2} of this expansion, are l**ying in an arithmetic series,** show that $\frac{1}{(21-n)!}+\frac{(n+2)(n+1)}{(23-n)!}=\frac{2(n+2)}{(22-n)!}$ Also find the values that n can take.

13. (*a*) Let $\mathbf{A}=\begin{pmatrix} a & 1 & b \\ -1 & -1 & a \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix} 4 & 1 \\ a & 0 \end{pmatrix}$; where $a, b \in \mathbb{R}$. It is given that $\mathbf{AB}^{\mathrm{T}}=\mathbf{C}$. Show that $a=2$ and $a=3$, and with these values for a and b, find $\mathbf{B}^{\mathrm{T}}\mathbf{A}$. Where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{B}$.

Write down $\mathbf{C}^{-1}$ and using it show that matrix $\mathbf{D}=4\begin{pmatrix} 1 & \lambda \\ 1 & -1 \end{pmatrix}$ such that $\mathbf{CD}=\mathbf{2I}+\mathbf{C}^2$, Where λ is a real contant that you should be determined.

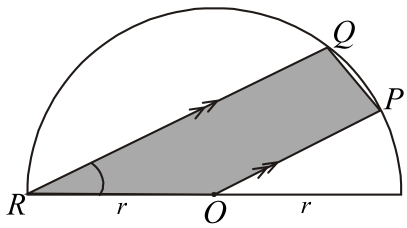
(*b*) Sketch in an Argand diagram, the locus C, of the points representing complex numbers z satisfing $|z|=1$. Now, let $\omega=a(\cos\theta+i\sin\theta)$ such that $a>0$ and $\frac{-\pi}{2}<\theta<\frac{\pi}{2}$.

Draw the locus of ω in the argand diagram. When ω lie on C. then show that $\omega-\frac{1}{\omega}$ is **purely imaginary** and the complex number $\omega-\frac{1}{\omega}$ lies between $-2i$ and $2i$ an the imaginary axis.

Futher more, when $\omega=\frac{1}{2}(1-\sqrt{3}i)$,

using the **moivre's theorem** show that $(1+\omega)^{24}+(1+\overline{\omega})^{24}=2(729)^2$

14. (*a*) The adjoining figure depicts, a quadrilateral $OPQR$ which is inscribed in a semi circle of radius r and centre O. Here if it is given that $O\hat{R}Q=\theta$; $0<\theta<\frac{\pi}{2}$ and $OP//RQ$ then show that the area S of the triangle quadri lateral is, $S=\frac{r^2}{2}(\sin\theta+\sin 2\theta)$. **Hence, deduce** that, the area S of the quadrilateral is maximum, when $\theta=\cos^{-1}\left(\frac{\sqrt{33}-1}{8}\right)$.



(*b*) Let, $f(x)=\dfrac{bx+c}{(x-a)^2}$ such that $a,\ b,\ c\in\mathbb{Z}-\{0\}$ for $x\in\mathbb{R}-\{a\}$. Show that, for any value of $a,\ b,\ c$ there exists a horizontal asymptote for the graph of $y=f(x)$, at $y=0$.

It there exist a vertical asymptotes at $x=1$, find the value of a.

Also, find the **integer values** of b and c, such that, $f(x)$, the first derivative of $f(x)$ is given by $f'(x)=\dfrac{3-x}{(x-a)^3}$; for $x\in\mathbb{R}-\{a\}$ **Hence,** by finding the co-ordinates of the turning point, write down the intervals on which $f(x)$ is decreasing and **the interval** on which $f(x)$ is increasing **Deduce** the nature of the turning point.

Further, if it is given that, $f''(x)$, the second derivative of $f(x)$ is $f''(x)=\dfrac{2(x+2c)}{(x-a)^4}$; for $x\in\mathbb{R}-\{a\}$ then, find the co-ordinates of point of inflection and write down the intervals on which $f(x)$ is concave down and the **interval** on which $f(x)$ is concave up.

Sketch the graph of $y=f(x)$ indicating the intercepts, asymptotes, turing point and point of inflection.

Now, let $A\equiv(2,\ 0)$. Find the equation of the tangent line drown to the graph of $y=f(x)$ at A and write down the co-ordinates of the point B of which the tangent meets the graph again.

15. (*a*) Find the **integers** A and B such that

$$27x^3-18x^2+12\equiv A(3x-2)^2+B(9x^2-12x+8)+A(3x-2)(9x^2-12x+8) \text{ for all } x\in\mathbb{R}.$$

Hence, write down $\dfrac{27x^3-18x^2+12}{(3x-2)^2(9x^2-12x+8)}$ in partial fractions and find

$$\int\frac{27x^3-18x^2+12}{(3x-2)^2(9x^2-12x+8)}\cdot dx$$

(*b*) Show that, $\dfrac{d\left[\ln\left(x+\sqrt{x^2-1}\right)\right]}{dx}=\dfrac{1}{\sqrt{x^2-1}}$. **Hence,** find $\displaystyle\int\frac{1}{\sqrt{x^2-1}}\cdot dx$.

Let $I=\int\sqrt{\tan x}\cdot dx$ and $J=\int\sqrt{\cot x}\cdot dx$ for $x\in\left(0,\ \dfrac{\pi}{2}\right)$. Using $\sin x-\cos x=t$ or otherwise show that, $I+J=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$, where C is a real constant. Using a suitable substitution, obtain a similar expression for $I-J$. **Hence,** find I and J.

Using $\displaystyle\int_a^b f(x)\cdot dx=\int_a^b f(a+b-x)\cdot dx$ where $a>b$ and $a,\ b\in\mathbb{R}$ **Hence,** deduce that,

$\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{\tan x}\cdot dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{\cot x}\cdot dx$ and show that, $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{\tan x}\cdot dx=\sqrt{2}\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$.

(*c*) **Using integration by parts** or otherwise, find $\int x\sin^{-1}x\cdot dx$

16. Let $l_1 \equiv kx - y + 1 = 0$; $k \in \mathbb{Z}^+$ and $l_2 \equiv x - 2y + 3 = 0$. $l_1 = 0$ and $l_2 = 0$ intersect the coodrinate axes. Let $S = 0$ is the circle which passes through the above mentioned points of intersections.

Find S and obtain the centre and the radius of the cricle. Also determine the value of k. Taking the same k value, find the point of intersection of $l_1 = 0$ and $l_2 = 0$. which is denoted by A.

Show that the **tangential chord** of the circle relative to A is $5x + y = 0$.

Find the equations of angle bisectors of $l_1 = 0$ and $l_2 = 0$ and **deduce** the equation of the obtused angle bisector for $k \in \mathbb{Z}^+$.

Show that there are two circles, in which the centre lie on the **obtused** angle bisector and $l_1 = 0$ and $l_2 = 0$ are the tangents also with radius $\sqrt{5}$ units.

Show also that one of the equations of a circle is $S_1 \equiv x^2 + y^2 - 4x - 1 = 0$ and find the other equation of the circle $S_2 = 0$.

Let $P \equiv (-1,\ 1)$, show the point P lies outside the circle $S_1 = 0$. Show also that the tangents drawn to the $S_1 = 0$ from the point P are $2x + y + 1 = 0$ and $x - 2y + 3 = 0$.

17. (*a*) Write down $\sin(A+B)$ in terms of $\sin A,\ \sin B,\ \cos A$ and $\cos B$. **Hence,** obtain a smilar expression for $\sin(A-B)$.

Deduce that $\sin\left(\frac{\pi}{2} - A\right) \equiv \cos A$ and $\sin(A-B)\cdot\sin(A+B) \equiv \sin^2 A - \sin^2 B$.

Further more **deduce** that $\cos 10^0 \cdot \cos 30^0 \cdot \cos 50^0 \cdot \cos 70^0 = \frac{3}{16}$

(*b*) Let $T(x) \equiv 2 + \sin x\left(\sqrt{3}\cos x - 2\sin x\right)$ for $x \in \mathbb{R}$. Find the real constants for A, B and α such that $T(x) \equiv A + B\cos(2x - a)$. where α ; $0 < \alpha < \frac{\pi}{2}$. Find the **range** of the function T for $\left[\frac{\pi}{6},\ 2\pi\right]$ and sketch the graph of $y = T(x)$ for the above domain, solve the equation $T(x) = 2$.

(*c*) In the usual notation, **state** the sine rule for a triangle ABC. Hence, **deduce** the cosine rule.

Show that $\frac{(a+b-c)(a+c-b)}{(a+b+c)(b+c-a)} \equiv \tan^2\left(\frac{A}{2}\right)$.